HOMEWORK 6

Due: 2026-06-18

1. Recall the several symbol classes introduced in Section 3.1 of Taylor: For $r \geq 0$,

- $p(x, \xi) \in \mathcal{A}^r S^m_{1,\delta}$ if
$$\|\partial_\xi^\alpha p(\cdot, \xi)\|_{C^r} \lesssim \langle \xi \rangle^{m-|\alpha|},$$
$$|\partial_x^\beta \partial_\xi^\alpha p(x, \xi)| \lesssim \langle \xi \rangle^{m-|\alpha|+\delta(|\beta|-r)}, \quad |\beta| > r,$$
- $p(x, \xi) \in \mathcal{A}_0^r S^m_{1,\delta}$ if
$$\|\partial_\xi^\alpha p(\cdot, \xi)\|_{C^{r+s}} \lesssim \langle \xi \rangle^{m-|\alpha|+\delta s}, \quad s \geq 0,$$
- $p(x, \xi) \in {}^r S^m_{1,\delta}$ if
$$\partial_x^\beta p(x, \xi) \in S^m_{1,\delta}, \quad |\beta| \leq r,$$
$$\partial_x^\beta p(x, \xi) \in S^{m+\delta(|\beta|-r)}_{1,\delta}, \quad |\beta| \geq r.$$

Check the inclusions
$$\mathcal{A}_0^r S^m_{1,\delta} \subseteq \mathcal{A}^r S^m_{1,\delta} \subseteq {}^r S^m_{1,\delta},$$
and also show that the three symbol classes are distinct.

2. Show the following fact we used in lecture, (3.5.25) in Taylor: For $r \in \mathbb{R}, p \in (1, \infty)$, $p(x, \xi) \in S^r_{1,0}$, and $X^r = W^{\frac{n}{p}+r,p}(\mathbb{R}^n)$,
$$P : X^r \to BMO.$$
Hint: We proved a part of the special case $p = 2$ while studying the Bahouri-Chemin-Danchin text. For general p, use kernel estimates.

3. Show the base case of the second part of (3.5.25) in Taylor, also revisited in (3.6.30): For $p(x, \xi) \in S^0_{1,0}$,
$$P : L^\infty \to BMO.$$

4. Write out the proof of the commutator estimate Proposition 3.6.B of Taylor (an outline is provided in the text): Given $p(x, \xi) \in S^0_{1,0}$, we have for $\sigma \in [0, 1]$,
$$\|P(fu) - fPu\|_{W^{\sigma,p}} \lesssim \|f\|_{C^1} \|u\|_{W^{\sigma-1,p}},$$
and for $\sigma > 1$,
$$\|P(fu) - fPu\|_{W^{\sigma,p}} \lesssim \|f\|_{C^1} \|u\|_{W^{\sigma-1,p}} + \|f\|_{W^{\sigma,p}} (\|u\|_{L^\infty} + \|Pu\|_{L^\infty}).$$

5. Recall the Weyl quantization,
$$(P^w(x, D)u)(x) = \int e^{i(x-y)\xi} p\left(\frac{x+y}{2}, \xi\right) u(y)\, dy d\xi.$$

Show that for $p, q \in S^m_{\rho,\delta}$,

$$P^w Q^w - Q^w P^w - S^w \in OPS^{2m-3(\rho-\delta)}_{\rho,\delta},$$

where $s(x,\xi)$ is given by the Poisson bracket,

$$s(x,\xi) = \{p,q\} = \sum_{i=1}^{n} \partial_{\xi_i} p \cdot \partial_{x_i} q - \partial_{x_i} p \cdot \partial_{\xi_i} q.$$