Spring 2026: Pseudodifferential Operators

HOMEWORK 4

Due: 2026-05-14

1. Recall the Besov space characterization of Zygmund spaces, $C^s_*(\mathbb{R}^n) = B^s_{\infty,\infty}(\mathbb{R}^n)$. Show that C^1_* is equivalent to the set of bounded continuous functions such that

$$\sup_{x,y\in\mathbb{R}^n} \frac{|u(x+y)+u(x-y)-2u(x)|}{|y|} < \infty.$$

2. Let $0 < \delta < \gamma < 1$. Let $p \in C^s S^m_{1,\delta}$ and define the smoothed symbol

$$p^\sharp := \sum_{\lambda=1}^{\infty} p_{\leq \lambda^{\gamma-\delta}}(x,\xi)\psi_\lambda(\xi)$$

where p is frequency-truncated in x. Show that

$$p^\sharp \in S^m_{1,\gamma}, \qquad p^b = p - p^\sharp \in C^s S^{m-(\gamma-\delta)s}_{1,\gamma}.$$

3. We focused on L^2 and Besov-type Sobolev embedding theorems in class. Prove the L^p-based Sobolev embedding theorems at the end of Appendix A.1 in Taylor. Precisely, show that

a) For $s < n/p$,

$$W^{s,p}(\mathbb{R}^n) \subseteq L^{\frac{np}{n-sp}}(\mathbb{R}^n),$$

b) for $s > n/p$,

$$W^{s,p}(\mathbb{R}^n) \subseteq L^\infty(\mathbb{R}^n),$$

c) for any $s \in \mathbb{R}$,

$$W^{s,p}(\mathbb{R}^n) \subseteq C^{s-\frac{n}{p}}_*(\mathbb{R}^n).$$