Spring 2026: Pseudodifferential Operators

HOMEWORK 2

Due: 2026-04-16

1. Show that the product estimate holds for the more classical (inhomogeneous) L^p Sobolev spaces. These are denoted by $W^{s,p}(\mathbb{R}^d)$, for $s \in \mathbb{R}$ and $p \in (1,\infty)$, with norm defined by

$$\|u\|_{W^{s,p}(\mathbb{R}^d)} := \|\langle D\rangle^s u\|_{L^p(\mathbb{R}^d)}.$$

More precisely, show that for $s \geq 0$ and $p \in (1,\infty)$, we have the estimate

$$\|uv\|_{W^{s,p}} \lesssim \|u\|_{L^\infty}\|v\|_{W^{s,p}} + \|v\|_{L^\infty}\|u\|_{W^{s,p}}.$$

2. Study endpoint cases for two of the theorems we discussed in class. More precisely:

a) Recall the paralinearization theorem (Theorem 2.92 in the text): Let $s, \rho > 0$, F smooth, ρ not an integer, $p, r_1, r_2 \in [1,\infty]$ such that $r_2 \geq r_1$, and

$$\frac{1}{r} = \min\left(1, \frac{1}{r_1} + \frac{1}{r_2}\right).$$

Then

$$\|F(u) - T_{F'(u)}u\|_{B^{s+\rho}_{p,r}} \lesssim_{\|u\|_{L^\infty}} \|u\|_{B^{\rho}_{\infty,r_2}}\|u\|_{B^{s}_{p,r_1}}.$$

b) Recall the commutator estimate (Lemma 2.99 in the text): Let f be smooth on $\mathbb{R}^d$ and homogeneous of degree m outside of the unit ball $B(0,1)$. Let $\rho \in (0,1), s \in \mathbb{R}$, and $p, p_1, p_2, r \in [1,\infty]$ with $1/p = 1/p_1 + 1/p_2$. Then

$$\|[T_a, f(D)]u\|_{B^{s-m+\rho}_{p,r}} \lesssim \|\nabla a\|_{B^{\rho-1}_{p_1,\infty}}\|u\|_{B^{s}_{p_2,r}}.$$

State parameters/spaces for which similar estimates hold when ρ is an integer in (a), and $\rho = 1$ in (b). You may switch to L^p or Sobolev spaces if appropriate (in fact, the textbook provides such an answer for (b)). Prove the new estimates.

3. Recall Proposition 0.2.B in Taylor: For $p(x,\xi) \in S^m_{1,\delta}$, the Schwartz kernel

$$K(x,y) = \int p(x,\xi)e^{i(x-y)\cdot\xi}\,d\xi$$

satisfies the estimate

$$|\partial^\beta_{x,y}K| \lesssim |x-y|^{-(n+m+|\beta|)}$$

provided that $m + n + |\beta| > 0$.

Complete the partial proof provided in the text: generalize to $\beta \neq 0$ and $p(x,\xi)$ depending on x.