Spring 2026: Pseudodifferential Operators

HOMEWORK 1

Due: 2026-04-02

1. Let $s_1 < s_2$. Show that both $\dot{H}^{s_1}(\mathbb{R}^d) \not\subseteq \dot{H}^{s_2}(\mathbb{R}^d)$ and $\dot{H}^{s_2}(\mathbb{R}^d) \not\subseteq \dot{H}^{s_1}(\mathbb{R}^d)$.

2. Show that:
(1) $\delta_0 \in H^s(\mathbb{R}^d)$ if $s < -\frac{d}{2}$,
(2) $\delta_0 \notin H^{-\frac{d}{2}}$,
(3) $\delta_0 \notin \dot{H}^s$ for any s.

3. Let ϕ be a Schwartz function. Let $s \in (0, \frac{d}{2})$ as in the Sobolev embedding theorem. Let $g : \mathbb{R}^d \to \mathbb{R}$ be given by

$$g(x) = |x|^{s-d}.$$

Show that

$$\phi = c_{d,s} \cdot (g * |D|^s \phi).$$

4. (new) Recall the counterexample to $\dot{H}^{\frac{d}{2}} \subseteq L^\infty$ in $d = 2$,

$$u(x) = \psi(x) \log(-\log |x|)$$

where ψ is smooth, $\psi \equiv 1$ on $B(0,1)$, and $\psi \equiv 0$ on $\mathbb{R}^2 \backslash B(0,2)$. Check from definition that this example is in BMO.